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THE PROBLEM OF WAVES OF DISCRETE SPECTRUM IN SHEAR FLOWS WITH SIGN - CONSTANT CURVATURE OF THE PROFILE*

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A Fourier transformation method is used to solve the Cauchy problem for a plane-parallel flow with a velocity profile with unchanging sign of the curvature. It is shown that in the case of an ideal fluid a perturbation forms (a residue at the pole of the integrand in the inverse Fourier transformation) in spite of the impossibility of the existence of a discrete eigenmode (Rayleigh's theorem), resembling a decaying eigenmode. This differs from the eigenmode in the fact that the residue in the critical layer is discontinuous. The total perturbation field appears to be continuous, since the discontinuity of the residue is compensated by a discontinuity in another component of the total field (non-harmonic with respect to time), namely by the integral along the edges of the cut. When the viscosity of the medium is taken into account, the pole of the integrand is displaced by a small amount proportional to $\nu^{1/3}$ (ν is the kinematic viscosity), the residue at the pole becomes a continuous function and corresponds to the decaying eigenmode for the Orr-Sommerfeld equation. When the viscous times are proportional to $\nu^{-1/3}$, the differences between the solutions of the Cauchy problem for the viscous and inviscid media become small.

According to Rayleigh's theorem, when the plane-parallel flow of an ideal fluid is such that the curvature of the velocity profile does not change its sign, no decaying or increasing eigenmodes are possible /1/. Neutral modes may exist in the flows with a curvature of constant sign only in the case when the curvature of the profile in the critical layer is equal to zero (such modes are easily computed in, e.g., piecewise-linear profiles /2/). When the curvature of the profile changes within the critical layer by an arbitrarily small amount, the mode either becomes increasing when the curvature of the whole profile becomes sign-alternating, or else it must, according to Rayleigh's theorem, vanish if the curvature remains sign-constant. Numerical solutions of the flows, of a viscous fluid with sign-constant profile curvature (in particular of the boundary layer), yield decaying modes which include modes whose phase velocity and decay coefficient cease to depend on the viscosity at large Reynolds numbers /3/, i.e. when the viscosity becomes vanishingly small the decaying modes which are forbidden by Rayleigh's theorem, remain present in the actual fluids with sign-constant curvature of the profile.

The problem of the disappearance of the eigenmodes in an ideal fluid for small deformations of the velocity profile is discussed below, together with the problem of the applicability of Rayleigh's theorem to actual fluids. Only the discrete-spectrum modes are dealt with (shear flows of an ideal fluid always contain, in addition to discrete spectrum modes, continuous spectrum mode which satisfy Rayleigh's equation in the generalized sense; the modes are neutral in principle, and their existence does not contradict Rayleigh's theorem /4-7/).

1. The eigenmode (wave) in plane-parallel flow with velocity profile $U(z)$ represents a harmonic perturbation satisfying Rayleigh's equation and boundary conditions (we shall consider, for simplicity, flows unbounded in z, since this does not affect the problems discussed here; for such flows the role of the boundary conditions will be played by the conditions for the perturbations to decay at infinity)

$$(\omega/k - U)(\varphi'' - k^2\varphi) + U''\varphi = 0 \tag{1.1}$$

$$\varphi \to 0, \quad |z| \to \infty \tag{1.2}$$

Here $\varphi(x, z, t) = \varphi(z)\exp(ikx - i\omega t)$ is a small perturbation in the stream function of the basic flow, k is the horizontal wave number, and $\varphi' = \partial\varphi/\partial z$.

According to Rayleigh's theorem /1/ there are no modes with complex ω if the function U'' is sign-constant. The theorem yields a result which seems strange at first sight. Let us

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consider the waves in a profile close to a piecewise-linear profile, with a single break:

$$U(z) = \begin{cases} \gamma_1 z + \sigma\gamma_1 k z^2/2 + o(\sigma), & z > 0 \\ \gamma_2 z + \sigma\gamma_1 k z^2/2 + o(\sigma), & z < 0 \end{cases} \tag{1.3}$$

$$\gamma_1 = U'(+0),\ \gamma_2 = U'(-0),\ \sigma = U''/(k\gamma_1)$$

Here γ_1, γ_2 are the flow velocity gradients above and below the break (suppose $\gamma_1 > \gamma_2 > 0$), σ is a small dimensionless parameter characterizing the curvature of the profile outside the break, and the break corresponds to a rapid, practically instantaneous change in the flow velocity gradient over a very small scale, smaller than all other scales encountered in this paper. The family of profiles described by Eq.(1.3) is shown in Fig.1a) $\sigma = 0$; b) $\sigma > 0$, and c) $\sigma < 0$.

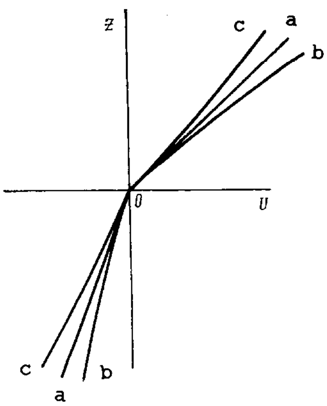

Fig.1

If the profile has zero curvature ($\sigma = 0$), then a unique discrete spectrum mode will exist in the flow (1.3):

$$\varphi = \exp(-k\,|\,z\,| + ikx - i\omega_0 t) \tag{1.4}$$

(to be specific we shall write $k > 0$) with the dispersion law

$$\omega_0 = (\gamma_1 - \gamma_2)/2 \tag{1.5}$$

Although the curvature of the profile does not change its sign when $\sigma = 0$: $U'' - (\gamma_1 = \gamma_2)\,\delta(z)$ and the existence of the wave (1.4), (1.5) does not contradict Rayleigh's theorem since it is neutral and the profile has zero curvature within its critical layer $z_c = \omega/(k\gamma_1)$.

When $\sigma < 0$, the curvature of the profile (1.3) is sign-alternating, and the eigenmode in such a profile becomes increasing, and can be found using the method of successive approximations, in the form of expansion in powers of σ. In particular, for the imaginary part of the frequency this method gives a linear dependence on σ for small σ (see Sect.2).

When $\sigma > 0$, the curvature of the profile (1.3) is sign-constant, and according to Rayleigh's theorem it has no eigenmodes.

Thus when $\sigma < 0$ (an unstable profile), an increasing mode will exist and its increment will decrease as the curvature decreases. The mode become neutral on changing to a piecewise-linear profile, and will vanish when the curvature changes further. Such a behaviour of the mode for small variations of the parameter σ is somewhat unusual, as if we had an oscillator in which, in the case of negative and zero friction, a corresponding increasing or neutral eigenoscillation existed and vanished on changing to an arbitrarily small positive value of the friction.

The determination of the eigenmodes in shear flows is related to the number of problems dealing with modes in laminar waveguides, which have been studied in detail in acoustics and radiophysics, and which reduce to solving equations of the type $\varphi'' + f(z, \omega)\,\varphi = 0$ with homogeneous boundary conditions (for (1.1) we have $f = U''/(\omega/k - U) - k^2$). In the initial problems of acoustic or electromagnetic waveguides the eigenmodes represent the residues at the poles of the solution spectrum /6, 8/. Thus the disappearance of a mode might lead us to expect that the pole in the solution spectrum of the initial problem will vanish when $\sigma > 0$. Below, we shall show that in shear flows the residue at the pole and the mode are not, in general, equivalent, and the disappearance of the mode does not imply the disappearance of the pole.

2. The reason for the disappearance of the mode when $\sigma > 0$ will become completely clear when the initial problem is solved. Let the velocity field be identical with (1.4) at the instant $t = 0$. At subsequent instants the field will satisfy the conditions (1.2) and the non-stationary Rayleigh equation, differing from (1.1) in that ω is replaced by $i\partial/\partial t$. To simplify the solution it will be convenient to replace the Cauchy problem by the equivalent problem with a source: let there be no perturbation at $t < 0$, and let a vertical external force act on the fluid at the instant $t = 0$, at the break in the profile

$$f_z = 2ik^{-2}\,\delta(z)\,\delta(t)\,e^{ikx} \tag{2.1}$$

In this case we can seek the solution separately in the upper region ($z > 0$) and the lower region ($z < 0$) where it is easier to obtain the particular solutions of Rayleigh's equation. We shall, however, demand that "matching" conditions hold at the break ($z = 0$), that φ is continuous, and that there is a pressure jump taking (2.1) into account

$$[p]_{z=0} = 2ik^{-2}\delta(t)\,e^{ikx},\ p = (ik^{-1}\partial/\partial t - U)\,\varphi' + U'\varphi \tag{2.2}$$

Here $[\ \ ]_{z=h}$ denotes a jump in the value of the function in the layer $z = h$, and p is the pressure.

We shall use the method of one-sided Fourier transformation in time (/9/, p.325)

$$\Phi(z,\omega)e^{ikx} = \int_0^\infty \varphi(x,z,t)e^{i\omega t}\,dt$$

The spectrum of the stream function $\Phi(z,\omega)$ satisfies, in the upper and lower region, the Rayleigh Eq.(1.1), conditions (1.2) and the "matching" conditions for the spectra

$$[\Phi]_{z=0} = 0,\quad [(\omega/k - U)\Phi' + U'\Phi]_{z=0} = -2i \tag{2.3}$$

We find the particular solutions of (1.1) in the upper region $\Phi_{+1,2}$ and lower region $\Phi_{-1,2}$ using the method of successive approximations and expansions in a small parameter σ

$$\begin{gathered}\Phi_{+1,2} = \Phi_{1,2}^{(0)} + \Phi_{+1,2}^{(1)} + \ldots,\ \Phi_{-1,2} = \Phi_{1,2}^{(0)} + \Phi_{-1,2}^{(1)} + \ldots \\ \Phi_{1,2}^{(0)} = e^{\mp kz},\quad \Phi_{+1,2}^{(1)} = \pm{}^1/_2\sigma\Phi_{1,2}^{(0)}F[\pm 2(kz - \omega/\gamma_1)] \\ \Phi_{-1,2}^{(1)} = \pm{}^1/_2\sigma\gamma_1\gamma_2^{-1}\Phi_{1,2}^{(0)}F[\pm 2(kz-\omega/\gamma_2)],\quad F(\zeta) = e^{\zeta}\,\mathrm{Ei}(-\zeta) - \ln\zeta\end{gathered} \tag{2.4}$$

"Matching" Φ_{+1} with Φ_{-2} and taking into account (2.3), we obtain the spectrum of the solution and write, using the inverse Fourier transformation, the solution of the problem in integral form

$$\begin{gathered}\varphi(x,z,t) = \frac{e^{ikx}}{2\pi}\int_\Gamma \frac{\Phi_{+1}(z,\omega)}{\Phi_{+1}(0,\omega)}\frac{e^{-i\varphi t}}{iW(\omega)}\,d\omega,\quad z>0 \\ W(\omega) = \omega_0 + {}^1/_2\omega k^{-1}(\Phi_{+1}'/\Phi_{+1} - \Phi_{-2}'/\Phi_{-2}) \approx \\ \omega_0 - \omega + {}^1/_2\omega\sigma\,[\exp(-2\omega/\gamma_1)\,\mathrm{Ei}(2\omega/\gamma_1) - \\ \gamma_1\gamma_2^{-1}\exp(2\omega/\gamma_2)\,\mathrm{Ei}(-2\omega/\gamma_2)] + O(\sigma^2)\end{gathered} \tag{2.5}$$

When $z<0$, we must replace $\Phi_{+1}(z,\ \omega)/\Phi_{+1}(0,\ \omega)$ by $\Phi_{-2}(z,\omega)/\Phi_{-2}(0,\omega)$ in (2.5). Henceforth, we shall only consider the region $z>0$.

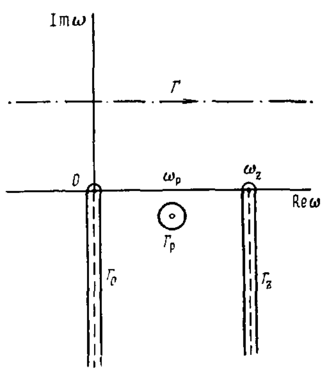

Fig.2

The contour Γ passes above all singularities of the integrand. The singularities are as follows: the logarithmic branch point $\omega = 0$ governed by the functions $\Phi_{+1}(0,\ \omega)$ and $W(\omega)$; the logarithmic branch point $\omega = \omega_z$ governed by the function $\Phi_{+1}(z,\ \omega)$ $(\omega_z = kU(z) = k\gamma_1 z + o(\sigma))$, and a pole $\omega = \omega_p$ governed by the zero of the function $W(\omega)$. The integrand in (2.5) is single-valued in the plane with cuts made from the branch points vertically downwards (Fig.2). We must choose in this plane the branches of the functions Ei and ln such that they take real values for negative and positive values of their arguments, respectively. When the branches are chosen in this manner, perturbation (2.5) will be identical with perturbation (1.4) at the instant $t=0$.

Taking the choice of branches into account, we find the pole ω_p using the method of successive approximations

$$\begin{gathered}\omega_p = \omega_r + i\omega_i;\ \omega_r = \omega_0 + {}^1/_2\sigma\omega_0 + [\exp(-2\omega_0/\gamma_1)\,\mathrm{Re\,Ei}(2\omega_0/\gamma_1) - \\ \gamma_1\gamma_2^{-1}\exp(2\omega_0/\gamma_2)\,\mathrm{Ei}(-2\omega_0/\gamma_2)] + O(\sigma^2) \\ \omega_i = -\sigma\pi\omega_0\exp(-2\omega_0/\gamma_1) + O(\sigma^2)\end{gathered} \tag{2.6}$$

When $t>0$, the integrand in (2.5) satisfies the conditions of Jordan's lemma in the lower half-plane. Let us deform the initial contour in the downward direction. We shall find that it will "hook" onto the singularities of the integrand and will split into three parts, namely a contour Γ_p, going around the pole, a contour Γ_0, going around the zero branch point along the cut edges, and a contour Γ_z, going around the branch point ω_z along the cut edges (Fig.2). We shall denote the results of integration along each contour by φ_p, φ_0, φ_z respectively.

We obtain the function φ_p using the theorem of residues, and write it in unique form, taking into account the choice of the branches $(z>0)$

$$\varphi_p = \exp(-kz + ikx - i\omega_p t)\,\{1 + {}^1/_2\sigma\,[\mathrm{Re}\,F(2kz - 2\omega_0/\gamma_1) - \tag{2.7}$$

$$\gamma_1\gamma_2^{-1} \exp(2\omega_0/\gamma_2)\, \mathrm{Ei}\,(-2\omega_0/\gamma_2) -$$
$$2\pi i\,[\exp(2kz - 2\omega_0/\gamma_1) - 1]\,\theta\,(z_c - z)] + o\,(\sigma)\}$$

The perturbation φ_p is harmonic in t and x, increases when $\sigma < 0$, and decreases when $\sigma > 0$.

When $\sigma < 0$, the perturbation φ_p satisfies the Rayleigh equation and conditions (1.2), and represents an increasing eigenmode. It can be excited in pure form, e.g. by a vertical volume force

$$f_z = -ik^{-3}(\varphi_p'' - k^2\varphi_p)\,\delta\,(t)$$

When $\sigma > 0$, the pole lies below the "mobile" branch point ω_z (it varies its position as z varies). When the point z passes through the critical layer z_c, a cut made from this branch point will intersect the pole, and the pole will be found at the other edge of the cut. Correspondingly, the residue of this pole will undergo a jump at the point $z = z_c$. The function φ_p has a break within the critical layer in terms of first order of smallness in σ, and a discontinuity in terms of second order of smallness in σ:

$$[\varphi_p']_{z=z_c} = \exp(-kz_c + ikx - i\omega_p t)\,\sigma k\pi i \qquad (2.8)$$
$$[\varphi_p]_{z=z_c} = -\exp(-kz_c + ikx - i\omega_p t)\,\sigma\omega_i\pi i\gamma_1^{-1}$$

When $\sigma > 0$, the perturbation φ_p will not satisfy Rayleigh's equation at only one point (within the critical layer). Therefore, it is not an eigenmode and cannot be excited in pure form without the application of external, constantly acting forces.

When $\sigma > 0$, the non-analyticity of the function φ_p in the critical layer is compensated by the non-analyticity of the function φ_z, so that their sum is analytic. We cannot calculate φ_z and φ_0 in explicit form, but we can show that for short times they are of the order of σ, and at $t \gg \gamma_1^{-1}$ their asymptotic form decreases as a power series $(z > 0)$:

$$\varphi_z = \begin{cases} \sigma\gamma_1^{-1}\exp(-kz + ikx)\,[it^{-1}\exp(-ikzt\gamma_1) + \\ \quad g\exp(-i\omega_p t)\,\mathrm{Ei}\,(igt)], & |gt| \lesssim 1 \\ \sigma t^{-2}\gamma_1^{-1}g^{-1}\exp(-kz + ikzt\gamma_1 + ikx), & |gt| \gg 1 \end{cases} \qquad (2.9)$$
$$g = \omega_r - \gamma_1 kz = \gamma_1 k\,(z_c - z)$$
$$\varphi_0 = -2\sigma t^{-2}\gamma_1^{-1}\gamma_2^{-1}\exp(-kz + ikx)$$

When calculating the asymptotic form (2.9) we used the fact that the integrand (2.5) decreases rapidly from the branch points along the cuts at $t \gg \gamma_1^{-1}$. Immediately after the force (2.1) has begun to act, the perturbations φ_z and φ_0 will be small compared with φ_p, but they will decay for $\sigma > 0$ more slowly than φ_p, and will be of the same order as φ_p when $t_\sigma \sim (\sigma\gamma_1)^{-1}$. When $t \gg t_\sigma$, the whole field will be largely determined by the perturbations φ_z and φ_0. Thus when $t \ll t_\sigma$, the total perturbation will differ little from harmonic with a time dependence $\exp(-i\omega_p t)$. When $\sigma > 0$ and $t \gg t_\sigma$, it will decrease as a power series. An analogous relationship for the decay of the perturbations not with time but along the horizontal coordinate was obtained in /10, 11/.

We note that the separation into harmonic and non-harmonic part is not unique. By changing the direction of the cut $(\omega_z, -i\infty)$, we also change the harmonic and non-harmonic part of the solution (but not their sum). In particular, the discontinuity of the residue z_c will be displaced (when the cut is purely vertical, the discontinuity will satisfy the condition $U(z_c) = \omega_r k^{-1}$). The closer the cut approaches the vertical line, the faster the integrand in (2.5) will decrease along the cut edges, and the asymptotic form (2.9) will be established for the non-harmonic part more quickly.

The results obtained can be applied to profiles of more general form, such as boundary layers, jets and wakes (although in these flows the determination of the poles is more difficult than in the case of the model profile (1.3)). Indeed, in the case of any real profile the perturbation spectrum will contain a "mobile" branch point (which will move along the axis $\mathrm{Re}\,\omega$ as z changes). The presence of this point is related to a particular feature of Rayleigh's equation, namely the fact that the coefficient of the principal derivative vanishes. The particular behaviour of the spectrum at this point is the same for all flow profiles: $U''(\omega - \omega_z)\ln(\omega - \omega_z)$. ($\omega_z$ is the frequency for which the layer with coordinate z is critical, i.e. $\omega_z k^{-1} = U(z)$). A cut from this point vertically downwards can intersect, at some z, the poles of the spectrum lying in the lower half-plane ω, and for these values of z the residues will have discontinuous and hence will not be eigenmodes (see the footnote in /12/). Thus Rayleigh's theorem holds for the modes, but not for the poles.

Nevertheless, if the curvature of the profile is small in the critical layer, the residue will "resemble" a weakly decaying mode since it depends harmonically on time and remains a dominant exponent of total perturbation a long time after the excitation ($t \sim k/U''(z_c)$). It should be expected that, in many waveguide problems (dealing with excitation by a force that is harmonic in time, with non-linear interactions, etc.), we can neglect the fact that the total perturbation differs from the residue, and regard the residue as a "fully valuable" mode.

3. Let us now solve the analogous problem for a viscous fluid, in which the force (2.1) acts on a viscous flow with profile (1.3). As in the case of an ideal fluid, we will use a Fourier transformation with respect to time. The spectrum of the solution $\Phi^\nu(z, \omega)$ will satisfy, in each region $z > 0$ and $z < 0$, the Orr-Sommerfeld equation

$$[(\omega/k - U)(\partial^2/\partial z^2 - k^2) + U'' - i\nu k^{-1}(\partial^2/\partial z^2 - k^2)^2]\,\Phi^\nu = 0 \tag{3.1}$$

conditions (1.2), and the "matching" conditions at the break in the profile. The latter conditions, in case of a viscous fluid, take the form

$$[\Phi^\nu]_{z=0} = \left[\frac{\partial}{\partial z}\Phi^\nu\right]_{z=0} = \left[\frac{\partial^2}{\partial z^2}\Phi^\nu\right]_{z=0} = 0, \quad \left[ikU' + \nu\frac{\partial^3}{\partial z^3}\right]_{z=0}\Phi^\nu = 2k \tag{3.2}$$

and follow from the conditions of continuity of both the velocity components and the vorticity, and from the conditions of a jump (2.2) in the pressure.

In the region $z > 0$ Eq.(3.1) has four particular solutions. We shall call two of these solutions the "inviscid" solutions $\Phi^\nu_{+1,2}$, and the other two the "viscous" solutions $\Phi^\nu_{+3,4}$. None of these solutions contains a branch point, and, unlike in the case of an ideal fluid, they are all analytic over the whole plane ω.

We shall write the asymptotic form of the inviscid solution for $|\delta_z| \gg R^{-1/3}$, where $R = \gamma_1/\nu k^2$ is the Reynolds number and $\delta_z = (\omega - \omega_z)/\gamma_1$, in the form of the sum of the inviscid and viscous terms

$$\Phi^\nu_{+1,2}(\omega, z) = \Phi_{+1,2}(\omega, z) + \Delta\Phi^\nu_{+1,2}(\omega, z) \tag{3.3}$$

Here $\Phi_{+1,2}$ are particular solutions of the Rayleigh equation described by expression (2.4), and the asymptotic form $\Delta\Phi^\nu_{+1,2}$ depends on the argument δ_z:

$$\Delta\Phi^\nu_{+1,2} = \begin{cases} {}^1/_3 i\sigma R^{-1}[2\exp(-2\delta_z)\,\mathrm{Ei}\,(2\delta_z) + \delta_z^{-2} - \delta_z^{-1}] + O(\sigma R^{-2}), & \omega \notin S_z \\ -{}^1/_2\sigma\Phi^\nu_{+3,4}, & \omega \in S_z \end{cases} \tag{3.4}$$

Here S_z is a region in which $-{}^5/_6\pi < \arg\delta_z < -{}^1/_6\pi$. This expression was obtained by a Fourier transformation with respect to the coordinate z. In the region $|\delta_z| \lesssim R^{-1/3}$ we can find the solutions $\Phi_{+1,2}{}^\nu$ using the method of small perturbations, and the solutions will be expressed in terms of the integrals of Hankel function of order ${}^1/_3$. In what follows, we shall only need an estimate of the viscous corrections in this region

$$\Phi^\nu_{+1,2}(\omega, z) = \Phi^{(0)}_{1,2}(z) + O(\sigma R^{-1/3}) \tag{3.5}$$

The asymptotic form of the viscous solutions for $|\delta_z| \gg R^{-1/3}$ was studied using the method of standard equations /13/, and can be written in the form

$$\Phi^\nu_{+3,4} \approx R^{-3/4}\delta^{-5/4}\exp\left(\mp R^{1/2}k\int Q^{1/2}dz\right)\left(1 \mp \frac{101}{48}iR^{-1/2}(-i\delta_z)^{-3/2} + O(R^{-1})\right), \tag{3.6}$$
$$Q(z) = i(kU - \omega)/\gamma_1 + R^{-1}$$

The solution Φ^ν_{+3} increases as $|\delta_z| \to \infty$ in the sector $-{}^5/_6\pi < \arg\delta_z < {}^1/_2\pi$, and the solution Φ^ν_{+4} in the sector $-{}^3/_2\pi < \arg\delta_z < -{}^1/_6\pi$. Thus the particular solutions $\Phi^\nu_{+1,2}$ increase as $|\delta_z| \to \infty$ in the sector S_z.

In the lower region $z < 0$ the particular solutions of Eq.(3.1) behave in exactly the same way, so we shall not write them out separately.

Using the expressions obtained for the asymptotic forms of the particular solutions, we shall solve the boundary value problem (3.1), (3.2), (1.2); we obtain

$$\Phi^\nu = \begin{cases} \Phi^\nu_{+1}(1 + O(\sigma) + O(R^{-1}))(\omega - \omega_p{}^\nu)^{-1}, & \omega \notin S_0 \\ (\Phi^\nu_{+1} + {}^1/_2\sigma\Phi^\nu_{+3})(1 + O(\sigma) + O(R^{-1}))(\omega - \omega_p{}^\nu)^{-1}, & \omega \in S_0 \end{cases} \tag{3.7}$$

$$\omega_p{}^\nu = \omega_p - (1 + i)\sqrt{2\nu k^2/\omega_0} \tag{3.8}$$

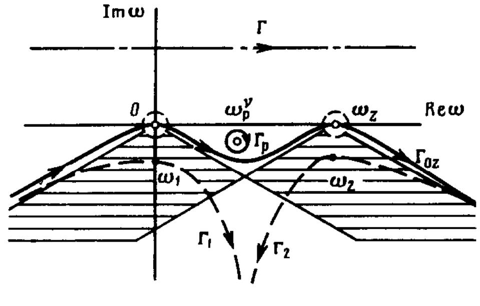

Fig.3

Here S_0 is a region in which $-{}^5/_6\pi < \arg\omega < -{}^1/_6\pi$, ω_p is a pole for the problem of the flow of an ideal fluid (see (2.6)). The spectrum of $z < 0$ in the region $z < 0$ behaves in the same way, so again we shall not write it out separately.

We shall write the solution of the initial Cauchy problem in the form of an inverse Fourier transformation:

$$\varphi^\nu(x, z, t) = (2\pi)^{-1} e^{ikx} \int_\Gamma \Phi^\nu(z, \omega) e^{-i\omega t} d\omega \tag{3.9}$$

When the fluid is viscous, the integrand does not satisfy the conditions of Jordan's lemma in the lower half-plane ω at $t > 0$. It increases as $\exp({}^2/_3 |\omega|^{3/2} \mathrm{Re}^{1/2})$ as $|\omega| \to \infty$ in the shaded regions in Fig.3. Formulas (3.4), (3.5) and (3.7) show that outside the sectors S_0 and S_z the viscosity corrections contribute little towards the spectrum Φ^ν.

Using representation (3.3), we shall split the field into two terms

$$\varphi^\nu = \varphi + \Delta\varphi^\nu$$

$$\varphi = (2\pi)^{-1} e^{ikx} \int_\Gamma \Phi_{+1} (1 + O(\sigma) + O(R^{-1}))(\omega - \omega_p{}^\nu)^{-1} e^{-i\omega t} d\omega \tag{3.10}$$

$$\Delta\varphi^\nu = (2\pi)^{-1} e^{ikx} \int_\Gamma \Delta\Phi_{+1}^\nu (1 + O(\sigma) + O(R^{-1}))(\omega - \omega_p{}^\nu)^{-1} e^{-i\omega t} d\omega \tag{3.11}$$

The integral (3.10) is analogous to the integral (2.5) which was discussed above. It yields a sum of fields $\varphi_p, \varphi_z, \varphi_0$, obtained for the ideal fluid, but the frequency of the harmonic part of the perturbation will be changed only slightly, according to (3.8).

Let us obtain an estimate for the integral (3.11). To do this we transform the initial contour into the contour Γ_{0z}, which passes along the outer periphery of the sectors S_0 and S_z (Fig.3). Integration over the small neighbourhoods of the points $\omega = 0$ and $\omega = \omega_z$, such that $|\omega/\gamma_1| \lesssim R^{-1/3}$ and $|\delta_z| \lesssim R^{-1/3}$ respectively yields, in accordance with estimate (3.5), a quantity of the order of $\sigma R^{-2/3}$. The integration over the remaining parts of the contour can be carried out approximately, and this will also yield a quantity of the order of $\sigma R^{-2/3}$, provided that $|kz| \gg R^{-1/3}$. Thus, taking into account the viscosity will lead everywhere, except in the neighbourhood of the break in the flow velocity profile, to the appearance of terms of the order of $\sigma R^{-2/3}$. When $\sigma \gg R^{-1/3}$, the terms will make a substantial contribution towards the general field only at the times $t_\nu = \gamma_1{}^{-1} R^{-1/3} = \nu^{-1/3} (kU')^{-2/3}$, at which $\varphi_0, \varphi_z \sim \sigma R^{-2/3}$ (analogous estimates were obtained in /6, 7/).

The asymptotic behaviour of the solution of the problem can be studied at $t \gg t_\nu$ using the saddle-point method. The integrand in (3.9) has two saddle points ω_1 and ω_2. When $t \gg t_\nu$, we obtain for them the following approximate expressions: $\omega_1 = -it^2\gamma_1{}^3 R^{-1}$, $\omega_2 = \omega_z + \omega_1$. Within these times the points in question will be displaced into the regions $-\mathrm{Im}\,\omega_{1,2}\gamma_1{}^{-1} \gg R^{-1/3}$, where we can use the asymptotic approximations for the particular solutions.

Let us transform the initial contour into the saddle contours Γ_1 and Γ_2 (Fig.3). If, in the course of deformation the contour intersects the pole ω_p, then we must, as usual, supplement the results of integrating over the saddle contours, with a residue at that pole. The saddle-point method yields

$$\varphi^\nu \sim \exp(-t^3/t_\nu{}^3) \tag{3.12}$$

Thus the character of decay of the perturbations changes over the times t_ν, from a power, to a exponential form $(\sim\exp(-t^3 t_\nu{}^{-3}))$. The first change in the form of decay of the perturbation from exponential $(\sim\exp(\omega_i t))$ to power $(\sim t^{-2})$ occurs at the time t_σ.

At the time $t_3 = t_\nu{}^{3/2} t_\sigma{}^{-1/2} = \sigma^{1/2}\gamma_1{}^{-1/2} k^{-1} \nu^{-1/2}$ the form of the perturbation decay changes for the third time to exponential ($\sim\exp(\omega_i t)$). Within these intervals the saddle points lie below the pole, while the residue at the pole just as at the times $t \ll t_\sigma$, becomes the most important part of the section

$$\varphi_p{}^\nu \approx \begin{cases} \Phi_{+1}^\nu(\omega_p{}^\nu, z)\exp(ikx - i\omega_p{}^\nu t), & kz \gg R^{-1/3} \\ [\Phi_{+1}^\nu(\omega_p{}^\nu, z) + {}^1/_2\sigma\Phi_{+3}^\nu(\omega_p{}^\nu, z)]\exp(ikx - i\omega_p{}^\nu t), & kz \lesssim R^{-1/3} \end{cases} \tag{3.13}$$

A threefold change in the asymptotic behaviour is possible only in the case when the effects connected with the curvature are "stronger" than the effects of viscosity, i.e., when $\sigma \gg R^{-1/3}$. Otherwise $t_\nu < t_\sigma$, and at $t \sim t_\nu$ we have the transition from φ_p to $\varphi_p{}^\nu$.

The perturbation $\varphi_p{}^\nu$ is formally identical with φ_p apart from terms of the order of σR^{-1}, apart from those z for which the pole falls within the sector S_z. We have near the critical layer $k\,|z - z_c| < \sqrt{3}\omega_i\gamma_1^{-1}$ and the break in velocity profile $|kz| \lesssim R^{-1/3}$, the viscosity makes a significant contribution towards $\varphi_p{}^\nu$. The perturbation $\varphi_p{}^\nu$ strongly oscillates in z, and the frequency and amplitude of the oscillations increase as the viscosity decreases.

The perturbation $\varphi_p{}^\nu$ satisfies the Orr-Sommerfiend equation for all z, as well as the conditions (1.2), and represents an eigenmode of viscous flow. When $\sigma \gg R^{-1/3}$, its dispersion and decay coefficient depend slightly on the viscosity (see (3.8)). The dispersion and form of this mode can be calculated approximaely (except in the neighbourhood of the critical layer), using the non-viscous approach discussed in Sect.2.

In conclusion we note that in a viscous fluid, unlike an ideal fluid, the profiles $U(z)$ cannot be arbitrary. In particular, the profile (1.3) must change with time so that the break in the velocity profile will be smoothed out. The change in the curvature of the profile $\Delta U''(z, t)$ is described by an equation which follows from the Navier-Stokes equations $d\Delta U''/dt = \nu\partial^2\Delta U''/\partial z^2$. From the point of view of the phenomena discussed here, it is important that such a change be small in the neighbourhood of the critical layer: $|\Delta U''(z_c, t)| \ll \sigma\gamma_1 k$. Computations show that the latter inequality breaks down at times of the order of $t_* \sim (\gamma_1/\gamma_2 - 1)^2 R/(\gamma_1 \ln \sigma^{-1})$, which are much longer than $t_\nu \sim R^{1/3}/\gamma_1$.

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